Stability of Internal Modes in Circularly Cylindrical MDH Equilibria *

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The stability of internal modes, i.e. modes which leave the plasma boundary unperturbed, is discussed for magnetohydrostatic equilibria in circularly cylindrical symmetry. Stability analysis can be performed analytically by expansion near the magnetic axis. Marginal stability conditions relating the pressure gradient and the shear are determined.

Let r, θ, z be cylindrical coordinates, p the pressure, and B_θ, B_z the poloidal and longitudinal magnetic fields. A necessary stability condition is Suydam's criterion [1]

$$p' + \frac{r}{8} B_z^2 \left(\frac{\mu'}{\mu}\right)^2 \geqq 0, \qquad (1)$$

where $\ldots' = \mathrm{d}/\mathrm{d}r \ldots$ and $\mu = B_\theta / r B_z$.

Suydam's criterion is, in general, not sufficient for internal modes, and the purpose of this paper is to investigate the modes which occur if it is satisfied and to find the profile conditions for which these modes are possible. To this end it suffices to consider the neighbourhood of the magnetic axis of a diffuse pinch; the stability problem can then be solved analytically. The neighbourhood of the magnetic axis is defined as follows: Modes in the interval $0 \leqq r \leqq R$ are considered, where R is chosen so small that

$$\varepsilon = R\,\mu_0 \ll 1, \quad \mu_0 = \mu(0).$$

Furthermore, it is assumed that the profile changes in $0 \leqq r \leqq R$ are small, described by

$$\frac{R^2 p'}{r B_z^2} \sim O(\varepsilon^2), \frac{R^2 \mu''}{\mu} \sim O(\varepsilon^2), \frac{R^4 p'''}{r B_z^2} \sim O(\varepsilon^4).$$

Correspondingly, it is required that the Taylor expansions

$$p' = -P_1 \mu_0^2 B^2 r - 2 P_3 \mu_0^4 B^2 r^3 + \cdots,$$
$$\mu = \mu_0 - S \mu_0^3 r^2 + \cdots, \quad B = B_z(0) \qquad (2)$$

exist, and that the dimensionless parameters P_1, P_3, and S (for shear) are finite.

In order that the cylinder be topologically equivalent to a torus, it is required that the disturbance posesses the period L in z, i.e. the corresponding wave number k must satisfy

$$k = 2\pi n / L, \quad n \text{ integer}.$$

According to [2] the energy principle reduces to the form

$$W = \int_0^{x_0} \left[f \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 + g\,\xi^2 \right] \mathrm{d}x, \qquad (3)$$

$$f = \frac{x^3 B_z^2 (n + m\iota)^2}{m^2 + n^2 x^2}, \quad x = \frac{2\pi r}{L}, \quad \iota = \frac{L\mu}{2\pi} \sim O(1), \quad x_0 = \frac{2\pi R}{L} \sim O(\varepsilon),$$

$$g = B_z^2 \left[\frac{2x^3 n^2}{m^2 + n^2 x^2} \frac{1}{x} \frac{\mathrm{d}p}{\mathrm{d}x} \frac{1}{B_z^2} + x(n + m\iota)^2 \frac{m^2 - 1 + n^2 x^2}{m^2 + n^2 x^2} + \frac{2 n^2 x^3}{(m^2 + n^2 x^2)^2} (n^2 - m^2 \iota^2) \right].$$

The case $m = 0$ is stable for $\varepsilon \to 0$, $B \neq 0$. The functional (3) therefore has to be discussed for $m = 1, 2, 3, \ldots$.

From Eq. (1) it is evident that $P_1 \leqq 0$ is necessary for stability. We prove that in the limit $\varepsilon \to 0$ the condition $P_1 < 0$ is sufficient for stability. For $P_1 \neq 0$ we have from Eqs. (2)

$$\frac{1}{x} \frac{\mathrm{d}p}{\mathrm{d}x} \frac{1}{B_z^2} \to -\iota_0^2 P_1, \quad \iota_0 = \iota(0).$$

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Suppose $\iota_0 + n/m \sim O(\varepsilon^\nu)$, $0 \leqq \nu < 2$, then

$$\frac{1}{B^2}\int_0^{x_0} f\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 \mathrm{d}x \to O(\varepsilon^{2\nu})\int_0^{x_0} x^3\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

$$\geqq O(\varepsilon^{2\nu})\int_0^{x_0} x\,\xi^2\,\mathrm{d}x \sim O(\varepsilon^{2\nu+1})\int_0^{x_0} \xi^2\,\mathrm{d}x\,. \qquad (4)$$

It is thus seen that the case $\nu = 0$ is always stable because the contribution of f to W is $O(\varepsilon)$, while the case $\nu > 0$, $P_1 < 0$ leads to the result that the only destabilizing term in g is $O(\varepsilon^{3+\nu})$, which completes the proof. Suydam's criterion is thus necessary and sufficient in leading order.

To proceed, we set $P_1 = 0$. Then

$$P_3 \leqq S^2/4$$

is obtained as necessary stability condition from Suydam's criterion, Equation (1). The above relation is shown as Curve I in Figures 1—4. Let us now suppose that $0 < \nu < 2$. The destabilizing terms in g are then small compared with the respective contribution of f. There is thus stability unless

$$\iota_0 + n/m = \iota_0^3\,Y, \quad Y \sim O(\varepsilon^2)\,. \qquad (5)$$

Then from

$$\frac{1}{x}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{1}{B_z^2} = -2P_3\,\iota_0^4\,x^2 + \cdots,$$

$$\iota = \iota_0 - S\,\iota_0^3\,x^2 + \cdots$$

as leading orders for f and g

$$\frac{1}{x}f = y\,B^2\,\iota_0^6\,(Y - S\,y)^2, \quad y = x^2\,,$$

$$\frac{1}{x}g = B^2\,\iota_0^6\,[-4P_3\,y^2 + (Y - S\,y)^2(m^2-1) - 4y(Y - S\,y)]$$

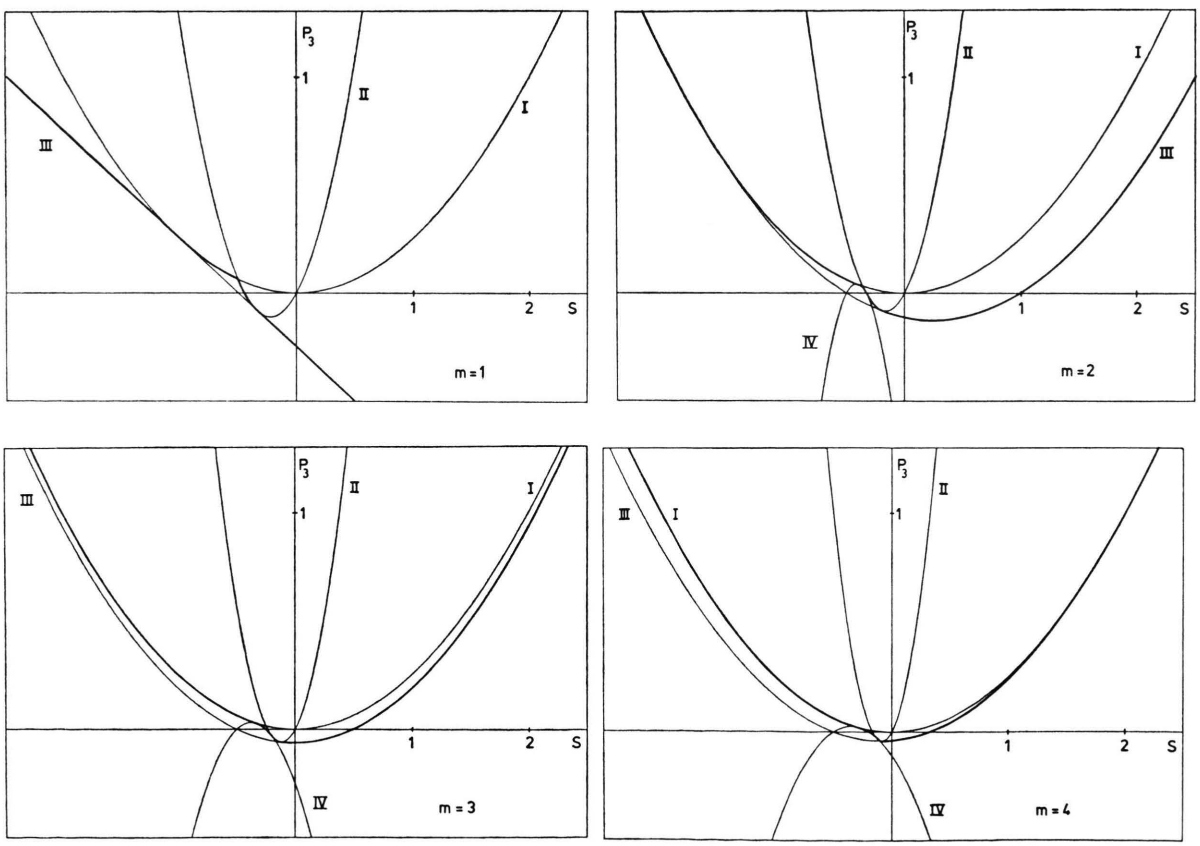

Fig. 1—4. Stability diagram of the diffuse linear pinch in terms of the pressure gradient as a function of the shear. I: $P_3 = S^2/4$, II: $P_3 = Q$, III: $P_3 = -S^2\beta_1(\beta_1+1)$, IV: $P_3 = -S^2\beta_2(\beta_2+1)$.

are obtained and

$$W = 2B^2 \iota_0^6 W_1,$$

$$W_1 = \int_0^{x_0^2} \mathrm{d}y \{y^2(Y - Sy)^2(\mathrm{d}\xi/\mathrm{d}y)^2 + [-P_3 y^2 + \tfrac{1}{4}(Y - Sy)^2(m^2-1) - y(Y - Sy)]\xi^2\}. \quad (6)$$

This is the main result which shows that the system is stable if and only if the functional W_1 is positive for all $Y \sim O(\varepsilon^2)$, $x_0^2 \sim O(\varepsilon^2)$.

Let us first consider $S=0$. For $Y \leqq 0$ the condition $P_3 \leqq 0$ is necessary and sufficient, the necessity following from $Y=0$. For $Y>0$, we introduce $z=y/Y$ and find as the most unstable case

$$Y^{-3} W_1 = W_2 = \int_0^\infty \mathrm{d}z \{z^2(\mathrm{d}\xi/\mathrm{d}z)^2 + [-P_3 z^2 + \tfrac{1}{4}(m^2-1) - z]\xi^2\}. \quad (7)$$

According to the functional (7) a test function ξ is called admissible if the normalization

$$\int_0^\infty (1+z^2)\xi^2 \mathrm{d}z = 1 \quad (8)$$

is possible.

The Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}z} z^2 \frac{\mathrm{d}\xi}{\mathrm{d}z} + [P_3 z^2 - \tfrac{1}{4}(m^2-1) + z]\xi = 0$$

for the functional W_2 can be solved in the form

$$\xi = z^{(m-1)/2} \exp\{-z/(m+1)\} \quad (9)$$

for $P_3 = -1/(m+1)^2$.

Since the solution (9) is admissible, it follows that

$$P_3 \leqq -1/(m+1)^2 \quad (10)$$

is necessary for stability. On the other hand, the functional (7) can be written in the form

$$W_2 = \int_0^\infty \mathrm{d}z \left\{z^2\left[\frac{\mathrm{d}\xi}{\mathrm{d}z} - \frac{1}{2}(m-1)\frac{\xi}{z} + \frac{\xi}{m+1}\right]^2 - \left[P_3 + \frac{1}{(m+1)^2}\right]\xi^2\right\},$$

which shows that the condition (10) is also sufficient.

Secondly let us consider $S \neq 0$.

a) $Y=0$. Here, we have for the most unstable case

$$\varepsilon^{-6} W_1 = W_3 = \int_0^\infty \mathrm{d}z \{z^4 S^2 (\mathrm{d}\xi/\mathrm{d}z)^2 + [-P_3 + \tfrac{1}{4}(m^2-1)S^2 + S] z^2 \xi^2\}, \quad (11)$$

where

$$y = \varepsilon^2 z$$

and the normalization condition (8) is imposed. The Euler-Lagrange equation for the functional (11) has the solutions

$$\xi = z^\varkappa, \quad \varkappa = -\frac{3}{2} \pm \frac{1}{S}\sqrt{-P_3 + Q},$$

$$Q = S\left(1 + \frac{9}{4}S\right) + \frac{1}{4}S^2(m^2-1).$$

These solutions are oscillatory, i.e. admissible, unless

$$P_3 \leqq Q. \quad (12)$$

Condition (12) is thus necessary. The form

$$W_3 = \int_0^\infty \mathrm{d}z \left[z^2\left(z\frac{\mathrm{d}\xi}{\mathrm{d}z} + \frac{3}{2}\xi\right)^2 S^2 + (-P_3 + Q) z^2 \xi^2\right]$$

shows that condition (12) is also sufficient. The relation $P_3 = Q$ is shown as curve II in Figures 1–4.

b) $Y \neq 0$. Here we set

$$y = -(Y/S)z$$

and see that there are three classes of modes. In the case A, $Y/S < 0$ there is no singular point in the integration interval. For $Y/S > 0$ there is a singular point at $z=-1$, and we distinguish between mode B, $-1 \leqq z \leqq 0$, and mode C, $-\infty \leqq z \leqq 1$. The stability of all three kinds of modes is therefore governed by the functional

$$\left|\frac{S^3}{Y^3}\right| W_1 = W_4 = \int_{-\infty}^{+\infty} \mathrm{d}z \left\{z^2(1+z)^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^2 + \left[\frac{P_3}{S^2}z^2 + \frac{1}{4}(m^2-1)(1+z)^2 + \frac{1}{S}z(1+z)\right]\xi^2\right\} \quad (13)$$

with the normalization

$$\int_{-\infty}^{+\infty} (1+z^2)\xi^2 \mathrm{d}z = 1. \quad (14)$$

The Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}z} z^2(1+z)^2 \frac{\mathrm{d}\xi}{\mathrm{d}z} \quad (15)$$

$$\left[-\frac{P_3}{S^2}z^2 + \frac{1}{4}(m^2-1)(1+z)^2 + \frac{1}{S}z(1+z)\right]\xi = 0$$

for the functional (13) can be solved in the form

$$\xi = |z|^\alpha |1+z|^\beta , \tag{16}$$

where

$$\alpha(\alpha+1) = \tfrac{1}{4}(m^2-1), \tag{17}$$

$$\beta = (1/2S - \alpha)/(\alpha+1) \tag{18}$$

if

$$P_3 = -S^2 \beta(\beta+1).$$

Case A ($z \geqq 0$): Here, a function of the form (16) is admissible if

$$\alpha > -\tfrac{1}{2}, \quad \alpha+\beta < -\tfrac{3}{2}.$$

This can be satisfied by choosing

$$\alpha = \alpha_1 = \tfrac{1}{2}(m-1),$$

$$\beta_1 = \left(\frac{1}{2S} - \alpha_1\right) \Big/ (\alpha_1+1) \tag{19}$$

as solution of Eqs. (17), (18) if the shear parameter S is restricted so that

$$\frac{1}{S} + \frac{1}{2}(m^2+m+4) < 0. \tag{20}$$

The choice (19) leads to the necessary condition

$$\begin{aligned} P_3 &\leqq -S^2 \beta_1(\beta_1+1) \qquad (21)\\ &= -\frac{1}{(m+1)^2}(1-mS+S)(1+2S)\\ &= \frac{1}{4}S^2 - \frac{1}{(m+1)^2}\left[1-\frac{1}{2}(m-3)S\right]^2\\ &= Q - \frac{1}{(m+1)^2}\left[1+\frac{1}{2}S(m^2+m+4)\right]^2 \end{aligned}$$

for stability. Relation (21) is shown as curve III in Figures 1—4. In order to prove sufficiency of condition (21) in the interval (20), we rewrite the functional (13) in the form

$$W_4 = \int_{-\infty}^{+\infty} \mathrm{d}z \tag{22}$$

$$\cdot \left\{ \left[z(1+z)\frac{\mathrm{d}\xi}{\mathrm{d}z} - \gamma(1+z)\xi - \delta z \xi \right]^2 \right\} + W_5 ,$$

$$\begin{aligned} W_5 = \int_{-\infty}^{+\infty} \mathrm{d}z\, \xi^2 \Bigg\{ & z^2 \left[-\frac{P_3}{S^2} + \frac{1}{4}(m^2-1) \right.\\ & \left. + \frac{1}{S} - \gamma^2 - \delta^2 - 2\gamma\delta - 3\gamma - 3\delta \right]\\ + z & \left[\frac{1}{2}(m^2-1) + \frac{1}{S} - 2\gamma^2 - 2\gamma\delta - 4\gamma - 2\delta \right]\\ & + \frac{1}{4}(m^2-1) - \gamma^2 - \gamma \Bigg\} \end{aligned}$$

and choose the constants γ, δ such that $\gamma = \alpha_1$, $\delta = \beta_1$. Then

$$W_5 = \int_0^\infty \mathrm{d}z\, \xi^2 z^2 \left[-\frac{P_3}{S^2} - \beta_1(\beta_1+1) \right], \tag{23}$$

which shows that W_4 is positive if condition (21) is satisfied.

Next we consider

$$\frac{1}{S} + \frac{1}{2}(m^2+m+4) \geqq 0 \tag{24}$$

and choose

$$\gamma = \tfrac{1}{2}(m-1), \quad \delta = -\tfrac{1}{2}(m+2),$$

yielding

$$W_5 = \int_0^\infty \mathrm{d}z\, \xi^2 \left\{ z^2 \left(-\frac{P_3}{S^2} + \frac{Q}{S^2} \right) + z \left[\frac{1}{S} + \frac{1}{2}(m^2+m+4) \right] \right\}. \tag{25}$$

This form shows that the mode A is stable in the region (24) if $P_3 \leqq Q$, i.e. below curve II.

Case B ($-1 \leqq z \leqq 0$): Here, a function of the form (16) is admissible if

$$\alpha > -\tfrac{1}{2}, \quad \beta > -\tfrac{1}{2}.$$

Again, we take the root (19) of Eq. (17) and restrict S to the interval

$$\frac{1}{S} - \frac{1}{2}(m-3) > 0. \tag{26}$$

In the same way as before it can be shown that condition (21) is necessary and sufficient for mode B if S is in the interval (26). If (26) is violated, we set

$$\gamma = \tfrac{1}{2}(m-1), \quad \delta = -\tfrac{1}{2}$$

and find

$$W_5 = \int_{-1}^{0} \mathrm{d}z\, \xi^2 \left[\left(-\frac{P_3}{S^2} + \frac{1}{4} \right) z^2 + z(z+1) \cdot \left(\frac{1}{S} - \frac{m}{2} + \frac{3}{2} \right) \right].$$

This mode is therefore stable outside the interval (26) below curve I.

Case C ($-\infty \leqq z \leqq -1$): Here, a function of the form (16) is admissible if

$$\beta > -\tfrac{1}{2}, \quad \alpha+\beta < -\tfrac{3}{2}.$$

This can be satisfied by choosing the other root of Eq. (17)

$$\alpha = \alpha_2 = -\tfrac{1}{2}(m+1)\,, \quad \beta_2 = \left(\frac{1}{2S} - \alpha_2\right) \Big/ (\alpha_2 + 1)$$

if both of the conditions

$$\frac{1}{S} < -\frac{1}{2}(m+3)\,, \tag{27}$$

$$\frac{1}{S} > -\frac{1}{2}(m^2 - m + 4) \tag{28}$$

are satisfied. This yields the necessary condition (curve IV in Figs. 1—4)

$$P_3 \leqq -S^2 \beta_2(\beta_2+1) \tag{29}$$
$$= -\frac{1}{(m-1)^2}(1 + mS + S)(1+2S).$$

Condition (29) is meaningless for $m=1$. However, for $m=1$ conditions (27) and (28) cannot be satisfied simultaneously. Sufficiency of condition (29) for the region (27), (28) is again shown by setting $\gamma = \alpha_2$, $\delta = \beta_2$ in the functional (22). Next, let us suppose condition (27) is violated. We then set

$$\gamma = -\tfrac{1}{2}(m+1), \quad \delta = -\tfrac{1}{2}$$

and find

$$W_5 = \int_{-\infty}^{-1} \mathrm{d}z\,\xi^2 \left[\left(-\frac{P_3}{S^2} + \frac{1}{4}\right) z^2 + z(z+1)\left(\frac{1}{S} + \frac{m}{2} + \frac{3}{2}\right)\right].$$

This means that mode C is stable below curve I in this case. Finally, let us suppose that condition (28) is violated. The choice

$$\gamma = -\tfrac{1}{2}(m+1)\,, \quad \delta = \tfrac{1}{2}(m-2)$$

then yields

$$W_5 = \int_{-\infty}^{-1} \mathrm{d}z\,\xi^2 \left\{\frac{1}{S^2}(-P_3 + Q)\,z^2 + \left[\frac{1}{S} + \frac{1}{2}(m^2 - m + 4)\right] z\right\},$$

showing that mode C is stable below curve II.

To summarize, stability has been shown below the heavy line in Figs. 1—4 and instability above. Note that, for $m=1$ the neutral curve has a break, while for $m>1$ the neutral curves are smooth due to the modes of class C. For the infinite cylinder only the result for $m=1$ applies, while for the finite cylinder the lowest value of m, which corresponds to relation (5), has to be picked to select the neutral curve. As an example, let us consider $\iota_0 \sim 3/2$ and $S=2$. Then $P_3 \leqq 5/9$ is sufficient for stability of internal modes in the limit $\varepsilon \to 0$. The results show that for $m>1$ there is qualitative agreement between Suydam's necessary criterion and the actual neutral curves.

The respective work for the self-consistent three-dimensional torus is in progress.

[1] B. R. Suydam, TID-7558, Controlled Thermonuclear Conference, held at Washington, D. C., Febr. 3-5, 1958.
[2] W. A. Newcomb, Ann. Physics **10**, 232 (1960).